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BMO-BOUNDEDNESS OF THE MAXIMAL OPERATOR FOR ARBITRARY MEASURES

BY

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ABSTRACT

We show that in the one-dimensional case the weighted Hardy–Littlewood maximal operator M_μ is bounded on $BMO(\mu)$ for arbitrary Radon measure μ, and that this is not the case in higher dimensions.

1. Introduction

Let μ be a nonnegative Radon measure on $\mathbb{R}^n$. For a bounded set $E \subset \mathbb{R}^n$ of positive μ-measure and a μ-locally integrable function f set

$$f_{E,\mu} = \frac{1}{\mu(E)} \int_E f(x) d\mu, \quad \Omega_\mu(f;E) = \frac{1}{\mu(E)} \int_E |f(x) - f_{E,\mu}| d\mu.$$

The Hardy–Littlewood and Fefferman–Stein maximal operators with respect to μ are defined by

$$M_\mu f(x) = \sup_{Q \ni x} |f|_{Q,\mu} \quad \text{and} \quad f^{\#}_\mu(x) = \sup_{Q \ni x} \Omega_\mu(f;Q),$$

respectively, where the supremum is taken over all cubes Q containing the point x such that $\mu(Q) > 0$. By a cube we mean an open cube with sides parallel to the axes.

If μ is a doubling measure (i.e., there exists a constant $c > 0$ such that $\mu(2Q) \le c\mu(Q)$ for all Q), then a classical result [13, p. 13] says that M_μ maps L^1_μ into weak-L^1_μ (in other words, M_μ is of weak type $(1,1)$) and L^p_μ into itself for $p > 1$. In the case $n = 1$ this result holds without any assumption on μ;

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when $n \geq 2$ there exist measures μ for which M_μ is not of weak type $(1,1)$ (see [11] and also [12, 16]).

In this paper we consider a question on the behaviour of M_μ on the space $BMO(\mu)$ for nondoubling measures. A function $f \in L^1_{\rm loc}(\mu)$ is said to belong to $BMO(\mu)$ if

$$\|f\|_{BMO(\mu)} \equiv \|f^{\#}_\mu\|_\infty < \infty.$$

We say that $f \in L^1_{\rm loc}(\mu)$ belongs to $BLO(\mu)$ (cf. [3]) if

$$\|f\|_{BLO(\mu)} \equiv \sup_Q (f_{Q,\mu} - \operatorname*{ess\,inf}_Q f) < \infty.$$

Note that $BLO(\mu) \subset BMO(\mu)$ and $\|f\|_{BMO(\mu)} \leq 2\|f\|_{BLO(\mu)}$.

It is well-known for doubling μ that if $f \in BMO(\mu)$ and $M_\mu f$ is not identically infinite, then $M_\mu f \in BLO(\mu)$, and

$$\|M_\mu f\|_{BLO(\mu)} \leq c_\mu \|f\|_{BMO(\mu)}. \tag{1.1}$$

In the case of Lebesgue measure, (1.1) was first proved by Bennett, DeVore and Sharpley [2] with the BMO-norm on the left-hand side; a BLO-improvement was obtained later in [1]. The method used in [1] works easily for any doubling measure. Different proofs of (1.1) were also given in [4, 7, 10, 15]. An attempt to generalize them to arbitrary measures leads only to measures satisfying the doubling condition. Also, all these proofs were essentially based (at least implicitly) on the John–Nirenberg inequality.

Observe that, as was shown in [8], the John–Nirenberg inequality for functions from $BMO(\mu)$ holds for a wide class of μ, namely, whenever $\mu(L) = 0$ for every hyperplane L, orthogonal to one of the coordinate axes. In the one-dimensional case this simply means that μ is continuous. Also, an example of singular μ was given in [8] for which the John–Nirenberg inequality does not hold.

The main result of this paper is somewhat surprising since it says that in the one-dimensional case (1.1) holds for any nonnegative Radon measure μ, even without the requirement of continuity of μ. Thus, we shall give a new proof of (1.1) that avoids both the doubling property of μ and the John–Nirenberg inequality. We also show that this geometrical phenomenon is not true in higher dimensions. The following theorem may be viewed as a BMO-counterpart of Sjögren's theorem from [11].

THEOREM 1.1: (i) *Let μ be a nonnegative Radon measure on $\mathbb{R}$. For any μ-locally integrable function f on $\mathbb{R}$ and for all $x \in \mathbb{R}$,*

$$M_\mu(M_\mu f)(x) \leq 40 M_\mu(f^{\#}_\mu)(x) + M_\mu f(x). \tag{1.2}$$

In particular, if $f \in BMO(\mu)$ and $M_\mu f < \infty$, then

$$\|M_\mu f\|_{BLO(\mu)} \le 40\|f\|_{BMO(\mu)}. \tag{1.3}$$

(ii) *For $n = 2$, there is an absolutely continuous measure μ for which M_μ is not bounded on $BMO(\mu)$.*

2. Auxiliary propositions

We shall use the following well-known properties of mean oscillations.

Proposition 2.1: *For any $f \in L^1_{\mathrm{loc}}(\mu)$,*

$$\Omega_\mu(|f|; E) \le 2\Omega_\mu(f; E), \tag{2.1}$$

$$\Omega_\mu(\max(f, 0); E) \le \Omega_\mu(f; E), \tag{2.2}$$

and

$$\Omega_\mu(f; E) \le 2 \inf_c \frac{1}{\mu(E)} \int_E |f(x) - c| d\mu. \tag{2.3}$$

We will also need the above-mentioned weak type $(1, 1)$ property of M_μ for arbitrary measures on $\mathbb{R}$.

Proposition 2.2: *Let μ be a nonnegative Radon measure on $\mathbb{R}$. Then for any $f \in L^1_\mu(\mathbb{R})$,*

$$\mu\{x \in \mathbb{R} : M_\mu f(x) > \alpha\} \le \frac{2}{\alpha}\|f\|_{L^1_\mu(\mathbb{R})} \quad (\alpha > 0). \tag{2.4}$$

This result was obtained in [11]; for a proof with constant 2 on the right-hand side of (2.4) see [5].

Proposition 2.3: *Let μ be a nonnegative Radon measure on $\mathbb{R}$. A nonnegative function $f \in L^1_{\mathrm{loc}}(\mu)$ belongs to $BLO(\mu)$ if and only if $M_\mu f - f$ belongs to L^∞, and*

$$\|f\|_{BLO(\mu)} = \|M_\mu f - f\|_\infty. \tag{2.5}$$

In the case of Lebesgue measure this was proved in [1, Lemma 2] for all $n \ge 1$; the proof was based on the notion of Lebesgue points of f. Due to Proposition 2.2, such approach in the one-dimensional case can be directly generalized to arbitrary measures. We omit details.

3. Proof of the main result

Proof of Theorem 1.1: We start with the first part of the theorem, so let $n = 1$. Assume that $f \geq 0$. Fix a point x, and let I be an arbitrary interval containing x. Let $R > 0$ be a number large enough such that $I \subset (-R, R)$. Let $y \in I$, and take any interval $J \subset (-R, R)$ containing y.

Suppose that $\mu(J) \geq \mu(I)/2$. Then setting $\widetilde{J} = I \cup J$, we have $\mu(\widetilde{J}) \leq 3\mu(J)$, and

$$\begin{aligned} f_{J,\mu} \leq |f_{J,\mu} - f_{\widetilde{J},\mu}| + f_{\widetilde{J},\mu} &\leq 3\Omega_\mu(f; \widetilde{J}) + M_\mu f(x) \\ &\leq 3f^{\#}_\mu(y) + M_\mu f(x). \end{aligned} \tag{3.1}$$

Assume now that $\mu(J) < \mu(I)/2$. Denote by L the union of I and all the intervals $J \subset (-R, R)$ with $J \cap I \neq \emptyset$ and $\mu(J) < \mu(I)/2$. It is clear that L is an interval. Also we easily have that $\mu(L) \leq 2\mu(I)$. Indeed, let $L = (a, b)$. For any $\varepsilon > 0$ there exist intervals $J_i, i = 1, 2$, such that $\mu(J_i) < \mu(I)/2$ and $(a + \varepsilon, b - \varepsilon) \subset J_1 \cup I \cup J_2$. Hence,

$$\mu((a + \varepsilon, b - \varepsilon)) < 2\mu(I) \quad \text{and} \quad \mu(L) = \lim_{\varepsilon \to 0} \mu((a + \varepsilon, b - \varepsilon)) \leq 2\mu(I).$$

Set

$$E_J = \{\xi \in J : |f(\xi) - f_{J,\mu}| \geq |f(\xi) - f_{L,\mu}|\}.$$

If $\mu(E_J) \geq \mu(J)/2$, then

$$\begin{aligned} f_{J,\mu} \leq 2 \inf_{\xi \in E_J} |f(\xi) - f_{J,\mu}| + f_{L,\mu} &\leq 4\Omega_\mu(f; J) + M_\mu f(x) \\ &\leq 4f^{\#}_\mu(y) + M_\mu f(x). \end{aligned} \tag{3.2}$$

If $\mu(E_J) < \mu(J)/2$, then $\mu(J \setminus E_J) \geq \mu(J)/2$, hence

$$\begin{aligned} f_{J,\mu} &\leq 2 \inf_{\xi \in J \setminus E_J} |f(\xi) - f_{L,\mu})| + f_{L,\mu} \\ &\leq 2m_\mu((f - f_{L,\mu})\chi_L)(y) + M_\mu f(x), \end{aligned} \tag{3.3}$$

where

$$m_\mu f(y) = \sup_{J \ni y} \sup_{E \subset J : \mu(E) \geq \frac{1}{2}\mu(J)} \inf_{\xi \in E} |f(\xi)|.$$

Unifying estimates (3.1), (3.2) and (3.3) yields

$$\begin{aligned} M^R_\mu f(y) &= \sup_{J \ni y, J \subset (-R,R)} f_{J,\mu} \\ &\leq 4f^{\#}_\mu(y) + 2m_\mu((f - f_{L,\mu})\chi_L)(y) + M_\mu f(x). \end{aligned} \tag{3.4}$$

It is easy to see that

$$\{y : m_\mu f(y) > \alpha\} \subset \{y : M_\mu \chi_{\{|f|>\alpha\}}(y) \geq 1/2\}.$$

Hence, by Proposition 2.2,

$$\begin{aligned}\|m_\mu f\|_{L^1_\mu} &= \int_0^\infty \mu\{y : m_\mu f(y) > \alpha\} d\alpha \\ &\leq 4 \int_0^\infty \mu\{y : |f(y)| > \alpha\} d\alpha = 4\|f\|_{L^1_\mu}.\end{aligned}$$

Therefore, using (3.4) and the fact that $\mu(L) \leq 2\mu(I)$, we have

$$\begin{aligned}\frac{1}{\mu(I)} \int_I M_\mu^R f(y) d\mu \leq & \frac{4}{\mu(I)} \int_I f_\mu^\#(y) d\mu + \frac{2}{\mu(I)} \|m_\mu((f - f_{L,\mu})\chi_L)\|_{L^1_\mu} + M_\mu f(x) \\ \leq & 4M_\mu(f_\mu^\#)(x) + 16\Omega_\mu(f; L) + M_\mu f(x) \\ \leq & 20M_\mu(f_\mu^\#)(x) + M_\mu f(x).\end{aligned}$$

Letting $R \to \infty$, and then taking the supremum over all I containing x, we obtain

$$M_\mu(M_\mu f)(x) \leq 20M_\mu(f_\mu^\#)(x) + M_\mu f(x). \tag{3.5}$$

This proves (1.2) for nonnegative f. The general case follows from (3.5) and (2.1). Next, (1.3) follows from (1.2) and from Proposition 2.3.

We turn to the second part of the theorem, suppose that $n = 2$. Let $Q_0 = \{(x, y) : 0 < x, y < 1\}$. Denote by Δ and Γ the open triangles bounded by the lines $y = x, y = 1/2, x = 0$, and $y = x, y = 0, x = 1$, respectively. Set now

$$d\mu = \chi_{Q_0 \setminus \Delta}(x, y) dx dy \quad \text{and} \quad f(x, y) = \big(\log(x/y)\big)\chi_\Gamma(x, y).$$

First we prove that $f \in BMO(\mu)$. Clearly it suffices to show that

$$\sup_{P \subset Q_0} \Omega_\mu(f; P) < \infty, \tag{3.6}$$

where the supremum is taken over all rectangles $P \subset Q_0$ with $\mu(P) > 0$. Observe that if $P \subset Q_0$, then $\mu(P) = |P \setminus \Delta|$, where $|\cdot|$ denotes the usual Lebesgue measure. Therefore one assumes that $P \setminus \Delta \neq \emptyset$.

Using (2.2) and the fact that $\log x \in BMO(0, 1)$, for every rectangle $P \subset Q_0$ we obtain

$$\Omega(f; P) \leq \Omega(\log(x/y); P) \leq 2\|\log x\|_{BMO(0,1)} \leq c \tag{3.7}$$

(we drop the subscript μ in the case of Lebesgue measure).

Take now an arbitrary rectangle $P \subset Q_0$ such that $P \setminus \Delta \neq \emptyset$. If $P \cap \Delta = \emptyset$, then by (3.7) we obtain

$$\Omega_\mu(f;P) = \Omega(f;P) \leq c.$$

Therefore, assume that $P \cap \Delta \neq \emptyset$. There are the following cases.

Case 1: Suppose that $P \cap \{y = x\} = \emptyset$. In this case we trivially have $\Omega_\mu(f;P) = 0$.

Case 2: Suppose that $P \cap \{y = x\} \neq \emptyset$ and $P \cap \{y = 1/2\} = \emptyset$. Let $\widetilde{P}$ be the smallest rectangle containing $P \setminus \Delta$. It is easy to see that $|\widetilde{P}| \leq 2|P \setminus \Delta|$. Therefore, by (2.3) and (3.7),

$$\Omega_\mu(f;P) \leq \frac{2}{|P \setminus \Delta|} \int_{P\setminus\Delta} |f(x,y) - f_{\widetilde{P}}|dxdy \leq 4\Omega(f;\widetilde{P}) \leq 4c.$$

Case 3: Suppose that $P \cap \{y = x\} \neq \emptyset$, $P \cap \{y = 1/2\} \neq \emptyset$ and $(1/2, 1/2) \in P$. If the point $(1/4, 1/4)$ does not belong to P, then $f(x,y) \leq \log 4$ for all $(x,y) \in P$, hence $\Omega_\mu(f;P) \leq 2\log 4$. If $(1/4,1/4) \in P$, then it is easy to see that $|P| \leq 4|P \setminus \Delta|$, and by (2.3) and (3.7) we have

$$\Omega_\mu(f;P) \leq \frac{2}{|P \setminus \Delta|} \int_{P\setminus\Delta} |f(x,y) - f_P|dxdy \leq 8\Omega(f;P) \leq 8c.$$

Case 4: Suppose that $P\cap\{y = x\} \neq \emptyset$, $P\cap\{y = 1/2\} \neq \emptyset$ and $(1/2, 1/2) \notin P$. Let $P_1 = P \cap \{y < 1/2\}$ and $P_2 = P \cap \{y \geq 1/2\}$. As in case 2, we take $\widetilde{P}_1$ to be the smallest rectangle containing $P_1 \setminus \Delta$. Denote by P_2' the rectangle P_2 translated until $\widetilde{P}_1$, and let $\widetilde{\widetilde{P}} = \widetilde{P}_1 \cup P_2'$. Observe that $\widetilde{\widetilde{P}}$ is a rectangle, and

$$|\widetilde{\widetilde{P}}| = |\widetilde{P}_1| + |P_2'| \leq 2|P_1 \setminus \Delta| + |P_2| \leq 2\mu(P).$$

Thus, applying again (2.3) and (3.7), we obtain

$$\begin{aligned}\Omega_\mu(f;P) &\leq \frac{4}{|\widetilde{\widetilde{P}}|}\left(\int_{P_2} |f - f_{\widetilde{\widetilde{P}}}|dxdy + \int_{P_1\setminus\Delta} |f - f_{\widetilde{\widetilde{P}}}|dxdy\right) \\ &= \frac{4}{|\widetilde{\widetilde{P}}|}\left(\int_{P_2'} |f - f_{\widetilde{\widetilde{P}}}|dxdy + \int_{P_1\setminus\Delta} |f - f_{\widetilde{\widetilde{P}}}|dxdy\right) \\ &\leq 4\Omega(f;\widetilde{\widetilde{P}}) \leq 4c.\end{aligned}$$

Combining all the cases considered above proves (3.6).

It remains to show that $M_\mu f \notin BMO(\mu)$. For $0 < \varepsilon < 1/4$ set $Q_\varepsilon = (0,\varepsilon) \times (1/2, 1/2+\varepsilon)$ and $\Delta_\varepsilon = \{(x,y) : 0 < y < x < \varepsilon\}$. Let us show that

$$M_\mu f(x,y) \geq \frac{1}{4} \log \frac{1}{\varepsilon}, \quad \text{for } (x,y) \in \Delta_\varepsilon \tag{3.8}$$

and

$$M_\mu f(x,y) \leq 16, \quad \text{for } (x,y) \in Q_\varepsilon. \tag{3.9}$$

Assuming for a moment (3.8) and (3.9) to be true, we note that they imply easily the desired result. Indeed, take a family of cubes Q'_ε such that $Q'_\varepsilon \cap Q_0 = P_\varepsilon$, where $P_\varepsilon = (0,\varepsilon) \times (0, 1/2+\varepsilon)$. By (3.8),

$$(M_\mu f)_{Q'_\varepsilon,\mu} \geq \frac{1}{\varepsilon^2 + \varepsilon^2/2} \int_{\Delta_\varepsilon} M_\mu f(x,y)dxdy \geq \frac{1}{12} \log \frac{1}{\varepsilon}.$$

Hence, by (3.9), for ε small enough we have

$$\begin{aligned}\Omega_\mu(M_\mu f; Q'_\varepsilon) &\geq \frac{2}{3\varepsilon^2} \int_{Q_\varepsilon} |M_\mu f(x,y) - (M_\mu f)_{Q'_\varepsilon,\mu}| dxdy \\ &\geq \frac{2}{3}\Big(\frac{1}{12} \log \frac{1}{\varepsilon} - 16\Big).\end{aligned}$$

Therefore,

$$\sup_{\varepsilon > 0} \Omega_\mu(M_\mu f; Q'_\varepsilon) = \infty, \tag{3.10}$$

which proves that $M_\mu f \notin BMO(\mu)$.

To show (3.8), take $(x,y) \in \Delta_\varepsilon$ and $x' < x$. Let Q be a cube such that $Q \cap Q_0 = (x', 1) \times (0, x')$. Then $(x,y) \in Q$, and

$$\begin{aligned}f_{Q,\mu} &= \frac{1}{x'(1-x')} \int_0^{x'} \int_{x'}^1 \log \frac{x}{y} dxdy \geq \frac{1}{4x'} \int_0^{x'} \log \frac{1}{2y} dy \\ &\geq \frac{1}{4} \log \frac{1}{x'} \geq \frac{1}{4} \log \frac{1}{\varepsilon},\end{aligned}$$

which proves (3.8).

We prove now (3.9). Take $(x,y) \in Q_\varepsilon$, and let Q be an arbitrary cube containing (x,y). We can assume that $Q \cap \{y = x\} \neq \emptyset$ and $Q \cap \{y = 1/2\} \neq \emptyset$, since otherwise $f_{Q,\mu} = 0$. If $Q \cap \{x = 0\} = \emptyset$, then $|Q \setminus \Delta| \geq 1/32$, and hence

$$f_{Q,\mu} \leq 32 \int_0^1 \int_0^x \log \frac{x}{y} dydx = 16.$$

Suppose that $Q \cap \{x = 0\} \neq \emptyset$, and let

$$A \equiv (Q \setminus \Delta) \cap \{y < 1/2\} = \{(x, y) : x \in (a, a + h), y \in (a, x)\}$$

for some $a \geq 0$ and $h > 0$. We can assume that $a + h < 1/2$, since otherwise $|Q \setminus \Delta| \geq 1/4$, and, as above we have $f_{Q,\mu} \leq 2$. Note that $|A| = h^2/2$. If $a = 0$, then

$$f_{Q,\mu} \leq \frac{1}{|A|} \int_0^h \int_0^x \log \frac{x}{y} dydx \leq 1.$$

Let $a > 0$ and $h < a$. Then

$$f_{Q,\mu} \leq \frac{1}{|A|} \int_a^{a+h} \int_a^x \log \frac{x}{y} dydx \leq 2 \log(1 + h/a) \leq 2 \log 2.$$

Let $0 < a \leq h$. Then

$$\begin{aligned} f_{Q,\mu} &\leq \frac{1}{|A|} \int_a^{a+h} \int_a^x \log \frac{x}{y} dydx \\ &= \frac{2}{h^2}(h^2/2 + ah - a(a + h) \log(1 + h/a)) \leq 3. \end{aligned}$$

All the cases considered above yield (3.9).

The theorem is proved. ∎

4. Concluding remarks

Remark 4.1: Inequality (1.2) is new even in the case of Lebesgue measure. Using similar arguments, one can show that for doubling μ this inequality holds for all $n \geq 1$ (of course with the constant depending on μ).

Remark 4.2: One can ask whether it is possible to "cancel" M_μ in both parts of (1.2) in order to get a stronger inequality

$$M_\mu f(x) \leq c f_\mu^{\#} f(x) + |f(x)|.$$

But this inequality is not true since it would imply that $BMO(\mu) = BLO(\mu)$ for positive f.

Remark 4.3: In proving the second part of Theorem 1.1 we actually have obtained that f belongs to the "strong" $BMO(\mu)$ (where the supremum is taken over all rectangular intervals instead of cubes). Therefore, in case of general μ and $n \geq 2$ even a stronger requirement that f belongs to the "strong" $BMO(\mu)$ does not imply that $M_\mu f$ belongs to the usual $BMO(\mu)$.

Remark 4.4: In the non-doubling situation several different generalizations of BMO are known. The space $BMO(\mu)$ considered here was studied in [8]. Tolsa [14] introduced another variant of BMO, the space $RBMO(\mu)$. In [6], a variant of BLO was introduced in that context, the space $RBLO(\mu)$, and for a kind of the maximal operator (defined by means of the so-called doubling cubes; see [14] for details) it was shown the boundedness from $RBMO(\mu)$ to $RBLO(\mu)$. Note that the proof of this fact follows the standard lines from [1], although with the use of the John–Nirenberg inequality for $RBMO(\mu)$ proved in [14].

Remark 4.5: It is well-known [3] that for doubling μ and for $0 < \delta < 1$ we have $M_\mu((M_\mu f)^\delta)(x) \le cM_\mu f(x)^\delta$. It was asked in [9, p. 2022] if this result holds for non-doubling μ. Using the same technique as in the proof of the first part of Theorem 1.1, one can show that in the one-dimensional case the answer is positive. We outline the proof briefly.

Proposition 4.6: *Let μ be a non-negative Radon measure. For any μ-locally integrable function f on $\mathbb{R}$ and for all $x \in \mathbb{R}$,*

$$M_\mu((M_\mu f)^\delta)(x) \le c_\delta M_\mu f(x)^\delta \quad (0 < \delta < 1).$$

Proof: We shall use the same notions as in the proof of Theorem 1.1. Let $x, y \in I$ and let $J \subset (-R, R)$ be an arbitrary interval containing y. If $\mu(J) \ge \mu(I)/2$, then

$$|f|_{J,\mu} \le 3|f|_{\tilde{J},\mu} \le 3M_\mu f(x).$$

Assuming $\mu(J) < \mu(I)/2$, we obtain

$$|f|_{J,\mu} \le M_\mu(f\chi_L)(y).$$

Therefore,

$$(M_\mu^R f(y))^\delta \le M_\mu(f\chi_L)(y)^\delta + 3^\delta M_\mu f(x)^\delta.$$

From this, by Proposition 2.2 and Kolmogorov's inequality,

$$\begin{aligned}\frac{1}{\mu(I)}\int_I (M_\mu^R f(y))^\delta d\mu &\le \frac{2^\delta}{1-\delta}\Big(\frac{1}{\mu(I)}\int_L |f| d\mu\Big)^\delta + 3^\delta M_\mu f(x)^\delta \\ &\le \Big(\frac{4^\delta}{1-\delta} + 3^\delta\Big) M_\mu f(x)^\delta.\end{aligned}$$

Letting $R \to \infty$ and taking the supremum over all I containing x completes the proof. ∎

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